Symmetric Monoidal ∞-categories.

Let Fin_* denote the category of finite pointed sets, we denote by $\langle n\rangle = \{0,1,2,\dots,n\}$ pointed by $\underline{0}$.
$(\langle n\rangle^\circ := \langle n\rangle \setminus \{0\})$.

Given an ordinary category C the data of a sym. mon. str. on C is that of a functor:

$$C^{\otimes} : Fin_* \to Cat \quad (\text{1-category of ordinary categories})$$

s.t.

- $C^{\otimes}(\langle 0\rangle) \simeq *$
- $C^{\otimes}(\langle 1\rangle) \simeq C.$
- $\forall\, n \geq 2$, let $e_i : \langle n\rangle \to \langle 1\rangle$ be the map of pointed sets determined by: $e_i(j) = \delta_{i,j}$.

$$C^{\otimes}(\langle n\rangle) \xrightarrow[\simeq]{C^{\otimes}(\prod_{i=1}^{n} e_i)} \prod_{i=1}^{n} C^{\otimes}(\langle 1\rangle).$$

The same definition work for ∞-cats.

__Def'n:__ Given an ∞-category $\mathcal{C}$ the data of a sym. monoidal str. on $\mathcal{C}$ is that of a functor:

$$\mathcal{C}^{\otimes} : Fin_* \to Cat_\infty, \quad \text{equivalently a coCartesian fibration:} \quad \mathcal{C}^{\otimes, Fin_*} \to Fin_*$$

s.t:

- $\mathcal{C}^{\otimes}(\langle 0\rangle) = *$, $\mathcal{C}^{\otimes,\mathrm{Fin}_*}_{\langle 0\rangle} \simeq *$
- $\mathcal{C}^{\otimes}(\langle 1\rangle) = \mathcal{C}$, $\mathcal{C}^{\otimes,\mathrm{Fin}_*}_{\langle 1\rangle} \simeq \mathcal{C}$.
- $\forall n \geq 2$ and $\langle n\rangle \to \langle 1\rangle^{\times n}$ the map given by $\coprod_{i=1}^{n} e_i$.

one has:

$$\mathcal{C}^{\otimes}(\langle n\rangle) \xrightarrow{\simeq} \prod_{n=1}^{\infty} \mathcal{C}^{\otimes}(\langle 1\rangle) \qquad \mathcal{C}^{\otimes,\mathrm{Fin}_*}_{\langle n\rangle} \xrightarrow{\simeq} \prod_{i=1}^{n} \mathcal{C}^{\otimes,\mathrm{Fin}_*}_{\langle 1\rangle}.$$

Rk: The usual functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is recovered as follows:

$$\mathcal{C}\times\mathcal{C} \xleftarrow{\simeq} \mathcal{C}^{\otimes}(\langle 2\rangle) \xrightarrow{\mathcal{C}^{\otimes}(\alpha)} \mathcal{C},$$ where

$$\alpha : \langle 2\rangle \to \langle 1\rangle, \quad \alpha(1) = \alpha(2) = 1.$$

Example: (i) Let $\mathcal{C}$ be an ∞-cat. w/ admits all finite products, in particular it has $*_{\mathcal{C}} \in \mathcal{C}$ a final object.
Let

$$\prod : \mathrm{Fin}_*^{op} \to \mathrm{Cat}_\infty.$$
$$\langle n\rangle \mapsto \mathrm{Fun}(\langle n\rangle, \mathcal{C}) \times_{\mathrm{Fun}(\langle 0\rangle,\mathcal{C})} *_{\mathcal{C}}.$$

This gives $\mathrm{Un}(\prod) \to \mathrm{Fin}_*$ a Cartesian fibration.

Claim: Since $\mathcal{C}$ has finite products. $\mathrm{Un}(\prod) \to \mathrm{Fin}_*$ is also a coCartesian fibration. and satisfies the condition to define a (Exercise!) sym. monoidal structure on $\mathrm{Un}(\prod)_{\langle 1\rangle} \simeq \mathcal{C}$.

Any example obtained this way is called a Cartesian sym. monoidal structure.

For instance, we could take $\mathcal{C} = \mathrm{Spc}$ or Cat_∞. (Same size argument here.)

(ii) (Lurie's tensor product.) Idea we want a tensor product between stable categories that is computable, i.e. commutes w/ colimits on each variable. Given $\mathcal{C}, \mathcal{D}$ & $\mathcal{E}$ st. ∞-cats we want:

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{E} \quad \text{to be determined by}$$

a functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ s.t. F preserves colimits in $\mathcal{C}$ and $\mathcal{D}$.

It turns out that this is easy to do. Consider $\mathrm{Cat}_\infty^{\times, \mathrm{Fin}_*} \to \mathrm{Fin}_*$ the coCartesian fibration corresponding to the Cartesian sym. mon. str. on Cat_∞.

<u>Def'ne - Prop:</u> Let $\mathrm{Cat}_\infty^{st, \otimes, \mathrm{Fin}_*}$ be the subcategory of $\mathrm{Cat}_\infty^{\times, \mathrm{Fin}_*}$ where:

- $\mathrm{Cat}_{\infty\, \langle 1 \rangle}^{st, \otimes, \mathrm{Fin}_*} \simeq \mathrm{Cat}_\infty^{st.} \subseteq \mathrm{Cat}_\infty$, i.e. we only consider complete st. ∞-categories.
- the functor $\mathrm{Cat}_\infty^{st, \otimes}: \mathrm{Fin}_* \to \mathrm{Cat}_\infty$ classified by $\mathrm{Cat}_\infty^{st, \otimes, \mathrm{Fin}_*}$ factors as follows:

$$\begin{tikzcd} \mathrm{Fin}_* \arrow[r] & \mathrm{Cat}_\infty^{st.} \arrow[d, hook] \\ \mathrm{Cat}_\infty^{st, \otimes} \arrow[r] & \mathrm{Cat}_\infty \end{tikzcd}$$

recall here we require functors to be continuous.

Let's briefly discuss maps between sym. mon. ∞-cats. For that we need a very useful notion:

A morphism $f: \langle n \rangle \to \langle m \rangle$ in Fin_* is said to be <u>idle</u> if $\forall j \in \langle m \rangle^\circ \quad |f^{-1}(j)| \leq 1$.

<u>E.g:</u> $\langle 1 \rangle \to \langle n \rangle$ is idle $\forall n \geq 1$.

$\langle 2 \rangle \xrightarrow[e_1, e_2]{} \langle 1 \rangle$ are idle but $\alpha: \langle 2 \rangle \to \langle 1 \rangle$ is <u>not</u>.

Def'n: A map of coCartesian fibrations:

$$\begin{tikzcd} \mathcal{C}^{\otimes,\mathrm{Fin}_*} \arrow[rr, "f"] \arrow[dr] & & \mathcal{D}^{\otimes,\mathrm{Fin}_*} \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd} \quad (\star)$$

is said to be:

- a symmetric monoidal functor. if it sends all coCart morphisms to coCart. morphisms.
- a right-lax sym. monoidal functor. if it sends idle coCartesian morphisms, i.e. morphisms whose image in Fin_* are idle, to coCartesian morphisms (necessarily idle too);
- a left-lax sym. monoidal functor: if one has a map

$$\begin{tikzcd} \mathcal{C}^{\otimes,\mathrm{Fin}_*^{op}} \arrow[rr, "f^{op}"] \arrow[dr] & & \mathcal{D}^{\otimes,\mathrm{Fin}_*^{op}} \arrow[dl] \\ & \mathrm{Fin}_*^{op} & \end{tikzcd}$$

between the Cartesian fibrations associated to ($\star$) which preserve idle Cartesian morphisms.

Rk: As usual sometimes we just say a functor $F: \mathcal{C} \to \mathcal{D}$ is sym. mon. (also right-lax sym. mon., etc.) if there exists a morphism between the coCart. fibrations.

- A justification of the names right-lax & left-lax above is the following:

FACT: let $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ be a pair of adjoint functors. then the following data are equivalent:

- a left-lax sym. monoidal str. on F;
- a right-lax. sym. monoidal str. on G.

Ex: Def'n: Given $\mathcal{C}^{\otimes}$ a sym. monoidal cat. a commutative algebra object in $\mathcal{C}^{\otimes}$ is the data of a right-lax sym. mon. fctor.

$A : \cancel{\emptyset}\ *^{\otimes, Fin_*} \to \mathcal{C}^{\otimes, Fin_*}$, where $*^{\otimes, Fin_*} = Fin_* \to Fin_*$.
classifies $*^{\otimes} : Fin_* \to Cat_\infty$.
$\langle n \rangle \mapsto *$

Example: ~~Let $Cat_\infty^{st.}$ denote the ∞-cat. of complete st. ∞-cats.~~

(i) Given any sym. mon. ∞-cat. $\mathcal{C}^{\otimes}$ the unit object is defined as:

$$\mathbb{1}_{\mathcal{C}} : * = \mathcal{C}^{\otimes}(\langle 0 \rangle) \to \mathcal{C}^{\otimes}(\langle 1 \rangle) = \mathcal{C}. \quad (\text{image of } \langle 0 \rangle \to \langle 1 \rangle).$$

there is an unique map.

It is easy to check that $\tilde{\mathbb{1}}_{\mathcal{C}}(\langle n \rangle) := \mathcal{C}^{\otimes}(\langle 0 \rangle \to \langle n \rangle)$ defines a right-lax sym. mon. fctor:

$$\tilde{\mathbb{1}}_{\mathcal{C}} : *^{\otimes, Fin_*} \to \mathcal{C}^{\otimes, Fin_*},$$

i.e. $\mathbb{1}_{\mathcal{C}}$ is a comm. alg. object in $\mathcal{C}^{\otimes}$.

Moreover, the coCartesian fibration corresponding to $\tilde{\mathbb{1}}_{\mathcal{C}}$ gives

$$\tilde{\mathbb{1}}_{\mathcal{C}}^{Fin_*} \to *^{\otimes, Fin_*} = Fin_*, \quad \text{i.e. if } \mathbb{1}_{\mathcal{C}} \in \mathcal{C} \text{ is}$$

some ∞-category, $\tilde{\mathbb{1}}_{\mathcal{C}}^{Fin_*}$ determines a sym. mon. str. on $\mathbb{1}_{\mathcal{C}}$.

(ii) Claim: the unit object in $\mathcal{P}r\,Cat_\infty^{st.}$ is $Spctr$.

In particular, $Spctr$ has a sym. monoidal structure.

~~All Comm alg. ...~~

RK: If one unpacks the definition of a comm. alg. here is the data one has:

The ~~...~~ inert morphisms $e_i : \langle 2 \rangle \to \langle 1 \rangle$ give maps

$$A(\langle 2 \rangle) \xrightarrow[A(e_i)]{} A(\langle 1 \rangle).$$

$A(e_i)$ being coCartesian mean $A(\langle 2 \rangle)$ is the tensor product of $A(\langle 1 \rangle)$ w/ itself.

The inert morphisms $\langle 1\rangle \xrightarrow{s_i} \langle 2\rangle$, $i=1,2$ & $s_i(1)=i$. give the left and right inclusion of unit.

Finally, since $\alpha: \langle 2\rangle \to \langle 1\rangle$ is not inert that means ~~given an elt $a \in A(\langle 1\rangle)$ one cannot find~~ an unique ~~$b \in A(\langle 2\rangle)$ s.t. $\alpha(b)=a$, i.e. a pair of elts multiplying to give a.~~

given "$a\otimes b \in$" $A(\langle 2\rangle)$, α does not determine the multiplication of a & b, i.e. to which elt of $A(\langle 1\rangle)$ they should be sent. This is extra data determined by $A(\alpha)$. as it should be. Rk: When $\mathcal{C}^{\otimes} \to Fin_*$ is a Cartesian $\otimes$-str. the data of $A: *^{Fin_*} \to \mathcal{C}^{\otimes, Fin_*}$ right-lax is equiv. to ☆,

(☆) $\bar{A}: Fin_* \to \mathcal{C} = \mathcal{C}^{\otimes}(\langle 1\rangle)$ s.t. $\forall \langle n\rangle \in Fin_*$ $\bar{A}(\langle n\rangle) \xrightarrow{\prod_{i=1}^{n} e_i} \prod_{i=1}^{n} \bar{A}(\langle 1\rangle)$ is an isom.

Similarly to how we dealt w/ sym. mon. structure we can do the case of only monoidal structures.

Def'n: Given an ∞-cat. $\mathcal{C}$ the data of a monoidal structure on $\mathcal{C}$ is a functor.

$$\mathcal{C}^{\otimes}: \Delta^{op} \to Cat_{\infty} \quad \text{s.t.}$$

- $\mathcal{C}^{\otimes}([1]) \simeq \mathcal{C}$
- $\mathcal{C}^{\otimes}([0]) = *$
- $\forall n \geq 2$, let $\rho_i: [1] \to [n]$ $\rho_i(0)=i$, $\rho_i(1)=i+1$ in Δ.

we require the induced map:

$$\mathcal{C}^{\otimes}([n]) \xrightarrow[\prod_{i=1}^{n} \mathcal{C}^{\otimes}(\rho_i)_{i=1}]{\simeq} \prod^{n} \mathcal{C}^{\otimes}([1])$$

to be an isomorphism.

We will write $\mathcal{C}^{\otimes, \Delta^{op}} \to \Delta^{op}$ for the induced coCartesian fibration.

Rk: In [HA, Def'n 2.4.2.1], Lurie uses a different ∞-operad to define monoidal structures: $Assoc^{\otimes}$, w/ $\mathcal{D}^{\otimes} \to Assoc^{\otimes}$ coCartesian + analogues of the conditions above.

The point is ([HA, Prop. 4.1.2.10]) Δ^{op} ($N(\Delta^{op})$ in loc. cit.) and $Assoc^{\otimes}$ gives the same notion of associative ∞-operad, though they are not equivalent as ∞-categories.

We can repeat the words to get definitions of strict, right-lax, left-lax monoidal functors & associative algebras in a mon. category. (Do it yourself!) We let $AssocAlg(\mathcal{C})$ denote the ∞-category of assoc. algs.

~~Rk: When $\mathcal{C}^{\otimes} \to Fin_*$ is a Cartesian~~

Notice we have a canonical functor: $\Delta^{op} \xrightarrow{oblv_{ord}} Fin_*$

$oblv_{ord}([n]) = \langle n \rangle$.

given $f : [m] \to [n]$ in Δ, i.e. a nondecreasing map.

$oblv_{ord}(f) : \langle n \rangle \to \langle m \rangle$.

$$i \mapsto \begin{cases} j, & \text{if } \exists j \in \langle m \rangle \text{ s.t. } f(j-1) < i \leq f(j) \\ 0, & \text{else.} \end{cases}$$

Thus, one has given $\mathcal{C}^{\otimes} : Fin_* \to Cat_\infty$ a sym. mon. category, one has $\mathcal{C}^{\otimes} \circ oblv_{ord} : \Delta^{op} \to Cat_\infty$ a mon. category.

Similarly, one has $CAlg(\mathcal{C}) \xrightarrow{oblv} AssocAlg(\mathcal{C})$.

We will say more about these next time.