

Symmetric Monoidal ∞ -categories.

Let Fin_* denote the category of finite pointed sets, we denote by $\langle n \rangle = \{0, 1, 2, \dots, n\}$ pointed by 0.
($\langle n \rangle^\circ := \langle n \rangle \setminus \{0\}$).

Given an ordinary category C the data of a sym. mon. str. on C is that of a functor:

$$C^{\otimes}: \text{Fin}_* \rightarrow \text{Cat} \quad (\text{1-category of ordinary categories})$$

s.t.

- $C^{\otimes}(\langle 0 \rangle) = *$

- $C^{\otimes}(\langle 1 \rangle) = C$.

- $\forall n \geq 2$, let $e_i: \langle n \rangle \rightarrow \langle 1 \rangle$ be the map of pointed sets determined by $e_i(j) = \delta_{ij}$.

$$C^{\otimes}(\langle n \rangle) \xrightarrow{\cong} \prod_{i=1}^n C^{\otimes}(\langle 1 \rangle).$$

The same definition work for ∞ -cats.

Def'n: Given an ∞ -category \mathcal{L} the data of a sym. monoidal str. on \mathcal{L} is that of a functor:

$$\mathcal{L}^{\otimes}: \text{Fin}_* \rightarrow \text{Cat}_{\infty}, \quad \text{equivalently a coCartesian fibration: } \mathcal{L}^{\otimes, \text{Fin}_*} \rightarrow \text{Fin}_*$$

s.t.

- $\mathcal{L}^{\otimes}(\langle 0 \rangle) = *$, $\mathcal{L}^{\otimes, \text{Fin}_*}(\langle 0 \rangle) = *$
- $\mathcal{L}^{\otimes}(\langle 1 \rangle) = \mathcal{L}$, $\mathcal{L}^{\otimes, \text{Fin}_*}(\langle 1 \rangle) = \mathcal{L}$
- $\forall n \geq 2$ and $\langle n \rangle \rightarrow \langle 1 \rangle^{x_b}$ the map given by $\prod_{i=1}^n e_i$.

one has:

$$\mathcal{L}^{\otimes}(\langle n \rangle) \xrightarrow{\cong} \prod_{i=1}^n \mathcal{L}^{\otimes}(\langle 1 \rangle) \quad \mathcal{L}^{\otimes, \text{Fin}_*}(\langle n \rangle) \xrightarrow{\cong} \prod_{i=1}^n \mathcal{L}^{\otimes, \text{Fin}_*}(\langle 1 \rangle)$$

Rk: The usual functor $\otimes: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ is recovered as follows:

$$\mathcal{L} \times \mathcal{L} \xleftarrow{\cong} \mathcal{L}^{\otimes}(\langle 2 \rangle) \xrightarrow{\mathcal{L}^{\otimes}(\alpha)} \mathcal{L}, \quad \text{where}$$

$$\alpha: \langle 2 \rangle \rightarrow \langle 1 \rangle, \quad \alpha(1) = \alpha(2) = 1.$$

Example: (i) let \mathcal{L} be an ω -cat. w/ admits all finite products, in particular it has $*$ $\in \mathcal{L}$ a final object.

let

$$\begin{aligned} \Pi: \text{Fin}_*^{\text{op}} &\rightarrow \text{Cat}_{\infty} \\ \langle n \rangle &\mapsto \text{Fun}(\langle n \rangle, \mathcal{L}) \times_{\text{Fun}(\langle 0 \rangle, \mathcal{L})} * \end{aligned}$$

This gives $U_n(\Pi) \rightarrow \text{Fin}_*$ a Cartesian fibration.

Claim: Since \mathcal{L} has finite products, $U_n(\Pi) \rightarrow \text{Fin}_*$ is also a Cartesian fibration, and satisfies the condition to define a (Exercise!!) sym. monoidal structure on $U_n(\Pi|_{\langle 1 \rangle}) = \mathcal{L}$.

Any example obtained this way is ^{called a} Cartesian sym. monoidal structure.

For instance, we could take $\mathcal{L} = \text{Spc}$ or Cat_{∞} . (same size argument here.)

(ii) (Lurie's tensor product.) Idea we want a tensor product between stable categories that is computable, i.e. commutes w/ colimits on each variable.

Given \mathcal{C}, \mathcal{D} & \mathcal{E} st. ∞ -cats we want:

$\mathcal{C} \otimes \mathcal{D} \rightarrow \mathcal{E}$ to be determined by a functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ s.t. F preserves colimits in \mathcal{C} and \mathcal{D} .

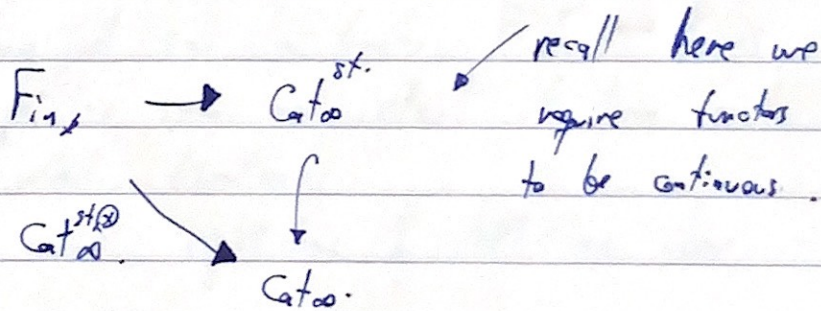
It turns out that this is easy to do. Consider $\text{Cat}_{\infty}^{x, \text{Fin}_x} \rightarrow \text{Fin}_x$ the coCartesian fibration corresponding to the Cartesian sym. mon. str. on Cat_{∞} .

Def'n - Prep: Let $\text{Cat}_{\infty}^{\text{st}, \otimes, \text{Fin}_x}$ be the subcategory of $\text{Cat}_{\infty}^{x, \text{Fin}_x}$

where: $\text{Cat}_{\infty}^{\text{st}, \otimes, \text{Fin}_x} \hookrightarrow \text{Cat}_{\infty}^{\text{st}} \subseteq \text{Cat}_{\infty}$, i.e.

we only consider $\text{st. } \infty$ -categories.

- the functor $\text{Cat}_{\infty}^{\text{st}, \otimes}: \text{Fin}_x \rightarrow \text{Cat}_{\infty}$ classified by $\text{Cat}_{\infty}^{\text{st}, \otimes, \text{Fin}_x}$ factors as follows:



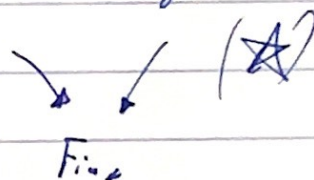
Let's briefly discuss maps between sym. mon. ∞ -cats. For that we need a very useful notion:

An isomorphism $f: \langle n \rangle \rightarrow \langle m \rangle$ in Fin_x is said to be idle if $\forall j \in \langle m \rangle \quad |f^{-1}(j)| \leq 1$.

E.g: $\langle 1 \rangle \rightarrow \langle 4 \rangle$ is idle $\forall n \geq 1$.

$\langle 2 \rangle \rightarrow \langle 1 \rangle$ are idle but $\alpha: \langle 2 \rangle \rightarrow \langle 1 \rangle$ is not.

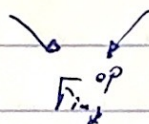
Def'n: A map of coCartesian fibrations: $f: \mathcal{L}^{\otimes, \text{Fin}_x} \rightarrow \mathcal{D}^{\otimes, \text{Fin}_x}$



is said to be:

- a symmetric monoidal functor. if it sends all coCart morphisms to coCart. morphisms.
- a right-lax sym. monoidal functor. if it sends idle coCartesian morphisms, i.e. morphisms whose image in Fin_x are idle, to coCartesian morphisms (necessarily idle too);

= a left-lax sym. monoidal functor: if one has a map $f^{\text{op}}: \mathcal{L}^{\otimes, \text{Fin}_x^{\text{op}}} \rightarrow \mathcal{D}^{\otimes, \text{Fin}_x^{\text{op}}}$ between the Cartesian fibrations associated to (\star) which preserve idle \otimes Cartesian morphisms.



Rk: \otimes As usual sometimes we just say a functor $F: \mathcal{L} \rightarrow \mathcal{D}$ is sym. mon. (or right-lax sym. mon., etc.) if there exists a morphism between the coCart. fibrations.

- A justification of the names right-lax & left-lax above is the following:

FACT: let $F: \mathcal{L} \rightleftarrows \mathcal{D} = \mathcal{G}$ be a pair of adjoint functors. then the following data are equivalent:

- a left-lax sym. monoidal str. on F ;
- a right-lax. sym. monoidal str. on G .

Ex:

Def'n: Given \mathcal{L}^{\otimes} a sym. monoidal cat. a commutative algebra object in \mathcal{L}^{\otimes} is the data of a right-lax sym. mon. functor.

$$A: \mathcal{C}^{\otimes, \text{Fin}x} \rightarrow \mathcal{L}^{\otimes, \text{Fin}x}, \quad \text{where } \mathcal{C}^{\otimes, \text{Fin}x} = \text{Fin}x \rightarrow \text{Fin}x.$$

classifiers $\mathcal{C}^{\otimes}: \text{Fin}x \rightarrow \text{Cato.}$

$$\langle u \rangle \mapsto x$$

Example: Let $\mathcal{Cato}^{\text{st.}}$ denote the ∞ -cat. of complete strat. ∞ -cats.

(i) Given any sym. mon. ∞ -cat. \mathcal{L}^{\otimes} the unit object is defined as:

$$\mathbb{1}_{\mathcal{L}}: x = \mathcal{L}^{\otimes}(\langle 0 \rangle) \rightarrow \mathcal{L}^{\otimes}(\langle 1 \rangle) = \mathcal{L}. \quad (\text{image of } \langle 0 \rangle \rightarrow \langle 1 \rangle).$$

there is an unique map.

It is easy to check that $\tilde{\mathbb{1}}_{\mathcal{L}}(\langle u \rangle) := \mathcal{L}^{\otimes}(\langle 0 \rangle \rightarrow \langle u \rangle)$ defines a right-lax sym. mon. functor:

$$\tilde{\mathbb{1}}_{\mathcal{L}}: \mathcal{C}^{\otimes, \text{Fin}x} \rightarrow \mathcal{L}^{\otimes, \text{Fin}x}$$

i.e. $\mathbb{1}_{\mathcal{L}}$ is a comm. alg. object in \mathcal{L}^{\otimes} .

Moreover, the coCartesian fibration corresponding to $\tilde{\mathbb{1}}_{\mathcal{L}}$ gives

$$\tilde{\mathbb{1}}_{\mathcal{L}}^{\text{Fin}x} \rightarrow \mathcal{C}^{\otimes, \text{Fin}x} = \text{Fin}x, \quad \text{i.e. if } \mathbb{1}_{\mathcal{L}} \in \mathcal{L} \text{ is some } \infty\text{-category, } \tilde{\mathbb{1}}_{\mathcal{L}}^{\text{Fin}y} \text{ determines a sym. mon. str. on } \mathbb{1}_{\mathcal{L}}.$$

(ii) Claim: the unit object in $\mathcal{Cato}^{\text{st.}}$ is Spectr.

In particular, Spectr. has a sym. monoidal structure.

ATD Comm. alg. \mathcal{L}^{\otimes}

RK: If one unpacks the definition of a comm. alg. here is the data one has:

The ~~see~~ unit morphisms $e_i: \langle 2 \rangle \rightarrow \langle 1 \rangle$ give maps

$$A(\langle 2 \rangle) \rightarrow A(\langle 1 \rangle).$$

$$A(e_i)$$

$A(e_i)$ being coCartesian mean $A(\langle 2 \rangle)$ is the tensor product of $A(\langle 1 \rangle)$ wr itself.

The α -invert morphisms $\langle 1 \rangle \rightarrow \langle 2 \rangle$, $i = 1, 2$ & $s_i(1) = i$ give the left and right inclusion of unit.

Finally, since $\alpha: \langle 2 \rangle \rightarrow \langle 1 \rangle$ is not invert that means given any object $a \in A(\langle 1 \rangle)$ one cannot find ~~an unique~~ $b \in A(\langle 2 \rangle)$ s.t. $\alpha(b) = a$, i.e. a pair of objects multiplying to give a .

given " $a \otimes b \in A(\langle 2 \rangle)$ " α does not determine the multiplication of a & b , i.e. to which object of $A(\langle 1 \rangle)$ they should be sent. This is extra data determined by $A(\alpha)$.

$\star) \bar{A}: \text{Fin}_* \rightarrow \mathcal{C}$
 $\mathcal{C}(\langle k \rangle)$. as it should be. Rk: When $\mathcal{C}^\otimes \rightarrow \text{Fin}_*$ is a Cartesian \otimes -str. s.t. $\forall \langle n \rangle \in \text{Fin}_*$ the data of $A: \text{Fin}_* \rightarrow \mathcal{C}^\otimes$ right-lax is equiv. to $\star)$

Similarly to how we dealt w/ sym. mon. structure we can do the case of only monoidal structures.

$$\bar{A}(\langle n \rangle) \rightarrow \prod_{i=1}^n \bar{A}(\langle 1 \rangle)$$

is an isom.

Def'n: Given an ω -cat. \mathcal{C} the data of a monoidal structure on \mathcal{C} is a functor.

$$\mathcal{C}^\otimes: \Delta^{op} \rightarrow \text{Cat}_\omega \quad \text{s.t.}$$

- $\mathcal{C}^\otimes(\langle 1 \rangle) = \mathcal{C}$
- $\mathcal{C}^\otimes(\langle 0 \rangle) = \mathbb{1}$
- $\forall n \geq 2$, let $p_i: \langle 1 \rangle \rightarrow \langle n \rangle$ $p_i(0) = i, p_i(1) = i+1$ in Δ .

we require the induced map: $\mathcal{C}^\otimes(\langle n \rangle) \xrightarrow{\cong} \prod_{i=1}^n \mathcal{C}^\otimes(\langle 1 \rangle)$ to be an isomorphism.

We will write $\mathcal{C}^{\otimes, \Delta^{op}} \rightarrow \Delta^{op}$ for the induced coCartesian fibration.

Rk: In [HA, Def'n 2.4.2.1] : Lurie uses a different ∞ -operad to define monoidal structures: $\text{Asoc}^{\otimes} \text{ w/ } \mathcal{J}^{\otimes} \rightarrow \text{Asoc}^{\otimes}$

Cartesian + analogues of the conditions above.

The point is ([HA, Prop. 4.1.2.10] Δ^{op} (N/A) is loc. cit.) and Asoc^{\otimes} gives the same notion of associative ∞ -operad, though \mathcal{J}^{\otimes} & \mathcal{L}^{\otimes} they are not equivalent as ∞ -categories.

We can repeat the words to get definitions of strict, right-lax, left-lax monoidal functors & associative algebras in a mon. category. (Do it yourself!) We let $\text{Asoc Alg}(\mathcal{C})$ denote the ∞ -category of assoc. algs.

~~Rk: When $\mathcal{C}^{\otimes} \rightarrow \text{Fin}_*$ is a Cartesian~~

Notice we have a canonical functor: $\Delta^{\text{op}} \xrightarrow{\text{oblvord.}} \text{Fin}_*$

$$\text{oblvord}([n]) = \langle n \rangle.$$

given $f: [m] \rightarrow [n]$ in Δ , i.e. a nondecreasing map.

$$\text{oblvord}(f) : \langle n \rangle \rightarrow \langle m \rangle.$$

$$i \mapsto \begin{cases} j, & \text{if } \exists j \in \langle m \rangle \text{ s.t. } f(j-1) < i \leq f(j) \\ 0, & \text{else.} \end{cases}$$

Thus, one has given $\mathcal{C}^{\otimes} : \text{Fin}_* \rightarrow \text{Cat}_{\infty}$ a mon. category, one has

$$\mathcal{C}^{\otimes} \circ \text{oblvord} : \Delta^{\text{op}} \rightarrow \text{Cat}_{\infty} \text{ a mon. category.}$$

Category.

Similarly, one has $\text{CAlg}(\mathcal{C}) \rightarrow \text{Asoc Alg}(\mathcal{C})$.

We will say more about these oblv next time.